High Performance Vibration Isolation Using Springs in Euler Column Buckling Mode.

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A new vertical suspension technique utilising the remarkable properties of column springs in Euler buckling mode allows the mass of suspension springs to be reduced towards their ultimate minimum, greatly increasing the resonant frequency of internal modes, and allowing near ideal vibration isolation to substantially higher frequencies than achievable conventionally.

1 Introduction

Vibration isolation consists in suspending a system in such a manner that motion of the support results in minimum motion of the isolated system. This is achieved by the mass of the isolated system being kept in place by soft linear restoring forces, thereby forming a harmonic oscillator with respect to the support. The softer the restoring force, the lower the resonant frequency, and the better the isolation obtained.

The stringent demands of gravitational wave detection has provided the motivation for the development of greatly improved vibration isolators[1]. In this paper we present a new use of elastic springs which allows vertical vibration isolation to achieve performance comparable to the horizontal vibration isolation of a simple pendulum. To explain the significance and benefits of the new technique it is necessary to discuss the intrinsic difficulties of conventional vertical isolator approaches.

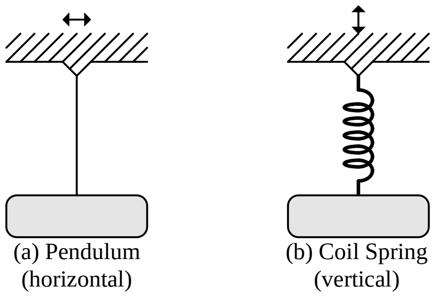

FIGURE 1. Typical horizontal and vertical isolation structures.

The simplest structure for isolating from *horizontal* motion is the pendulum as shown in figure 1(a), together with its isolation performance being the dashed curve in figure 2. A 25cm pendulum has a resonant frequency of approx 1Hz. Below resonance the isolated mass moves with the same amplitude as the support - and thus the transfer function has a magnitude of 1.0. At resonance the pendulum amplifies vibration by a factor determined by its Q-factor (typically several hundred) although this motion can normally be damped to greatly reduce this amplification. Above resonance the transfer function has a slope of -2 (on a log-log plot) so that isolation improves with frequency squared. Thus at 10Hz for instance, the transfer function is approximately 0.01 or 1% (meaning that for 10mm motion of the support, the isolated mass only moves 0.1mm). If the support shakes at higher frequencies, or if the pendulum is made longer thereby lowering the resonant frequency, then the isolation is better still.

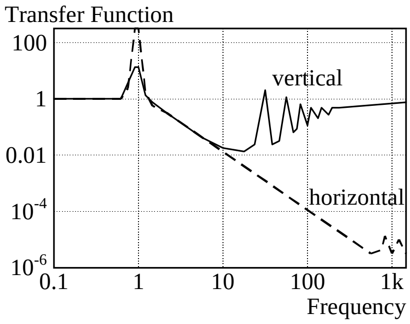

FIGURE 2. Typical transfer functions of horizontal and vertical structures.

In our laboratory we have demonstrated synthetic pendulum structures with very long periods that have achieved a transfer function with near ideal performance over almost 3 decades of frequency comparable to the dashed transfer function in figure 2 but at much lower frequency [2,3]. For 1mm applied vibration this corresponds to just 10nm amplitude at the optimum isolation frequency.

Vertical vibration isolation presents more problems than horizontal isolation because the vibrational motion (between the support and the isolated mass) in the presence of gravity requires the dynamic storage of significant amounts of energy (mgh) to momentarily absorb this motion. This energy storage is often provided in the form of mechanical springs. One of the simplest structures for isolating from vertical motion is the coil spring suspension shown in figure 1(b). In order for a *linear* spring (ie displacement $\propto$ force) such as this to achieve the same resonant frequency as a pendulum, it must be stretched under load (beyond its relaxed length) by the same length as the equivalent pendulum. Thus if a relaxed 10cm long coil spring is loaded with mass so that it becomes 35cm long, then the mass will oscillate with a frequency of 1Hz.

Most high performance vertical vibration isolators operate in a regime where the amplitude of vibration is very small compared with their extension under load. Under vibration only this small amount of dynamic energy is exchanged in and out of the spring while a large amount of energy remains statically in the spring due to its initial extension under load. The key point is that the total energy stored in the spring is much larger than the dynamic energy storage required in operation, as figure 3 illustrates.

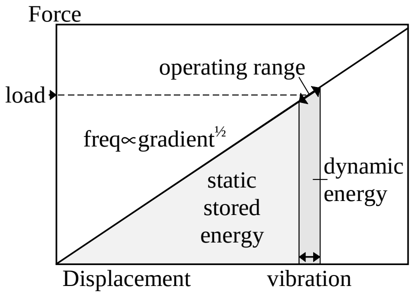

FIGURE 3. Force - displacement plot for a simple spring vertical isolator.

The large static energy storage necessitates a large mass of elastic material - proportional to the total static and dynamic energy to be stored. (In a pendulum on the other hand there is in principle no static energy and the dynamic energy is stored as gravitational potential energy - not requiring elastic material with mass) The disadvantage of the large mass requirement is that it supports low frequency *internal* resonant modes of the spring (eg "surging" in coil springs). These resonances have a large effective mass which strongly couple vibration between the support and the isolated mass at these internal resonant frequencies. Because the internal mode density increases with frequency, performance is strongly degraded as is illustrated in figure 2.

The same effect also occurs in the pendulum, but because the suspension fibre can be made so much less massive than the spring, its first internal (violin string mode) resonant frequency is much higher. In addition it couples much less energy due to the large ratio of the suspended mass to the fibre mass (the coupling depends also on its Q-factor).

2 Advanced Vertical Isolation

There are three main areas to consider in order to alleviate the spring mass problem inherent in vertical vibration isolators:- (1) ensure that the entire volume or mass of spring material is usefully storing energy by being stressed to its limit, (2) redistribute the mass of the spring to minimise its velocity and thus the kinetic energy of any internal mode motion, and (3) reduce the static energy and mass of the spring while keeping a low resonant frequency by producing a non-linear force vs displacement relationship.

The first of these areas only offers a small gain (a maximum of 2 for torsion and 3 for bending) and is rarely considered. A simple example would be to wind a coil spring from a tube rather than from solid material. This removes the central mass which is scarcely stressed and does not store much energy. This would be difficult but not inconceivable to apply to bending (one can imagine filling a central cavity with some lightweight and almost incompressible liquid!).

However when a spring is only stressed in one polarity, then in principle it is possible to preset initial stresses within the material so that the entire volume is stressed to its limit at some maximum deflection. For material loaded in torsion this is commonly called "scragging" (from "to wring the neck of"!). Consider a torsion bar which is plastically twisted well beyond its initial yield point and then released. The internal layers will not be able to relax with the amount of untwist that relaxes the surface layer, and so will strain the surface layers in the reverse direction until the torque from the internal layers is balanced. It is apparent that in order to twist such a bar again to its yield point, it must be twisted considerably further than before - initially to undo the reverse surface strain, and then the same as before to reach yield. Since the spring-rate is unaltered, such a torsion bar (which may be formed into a coil spring) can store considerably more energy for one direction of twist than it could before. The equivalent approach for flat springs is to bend them well past yield in the loaded direction. An example of this is curved cantilever blades and we believe this technique can also result in reduced creep at high stress levels[4].

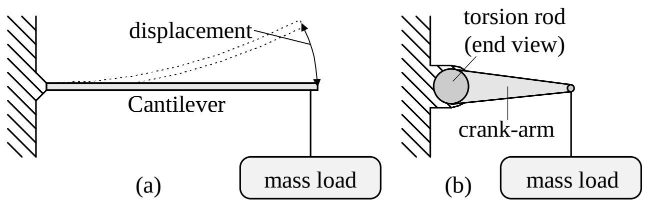

FIGURE 4. Concentrating the energy storage and thus mass to low velocity regions.

Examples of better distribution of the mass are shown in figure 4. The cantilever blade[5] in 4(a) with constant stress over its surfaces is in the shape of a triangle (top view) with the fixed attachment being the wide base of the triangle and the suspending tip being the apex of the triangle. This puts most of the mass near the fixed attachment where motion is minimum, thus achieving higher internal mode frequencies. Probably the best that can be achieved is the torsion rod suspension[6] shown in 4(b). Here the crank-arm can be made very rigid (to have very high internal mode resonances) and the spring material is all located very close to the fixed centre of rotation.

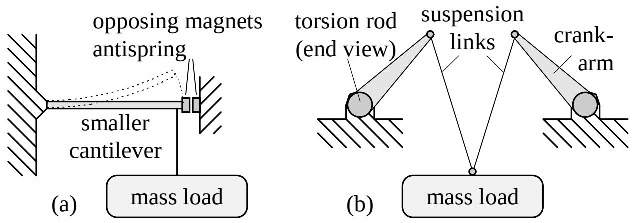

FIGURE 5. Anti-spring and non-linear techniques allow much smaller spring mass to achieve low resonant frequencies.

Examples of reducing the static energy and mass stored in the spring using non-linear spring techniques are shown in figure 5. The cantilever in 5a is fitted with magnets which strongly repel and try to drive the mass away from the normal operating position[7]. This is an anti-restoring force or "anti-spring" which when added to the normal restoring force of the spring produces a region of reduced gradient on the force displacement curve as is illustrated in figure 6. The main resonant frequency of an isolation system (eg: 1Hz in figure 2) is determined by the (square root of the) spring-rate which is the gradient of the force-displacement characteristic. By operating in the flattened region of the curve, a much lower resonant frequency is obtained than would normally be the case for the static displacement and energy storage employed. The torsion-crank suspension in 5b achieves a similarly flattened region in its force-displacement characteristic by its geometry and has been fully described previously [8].

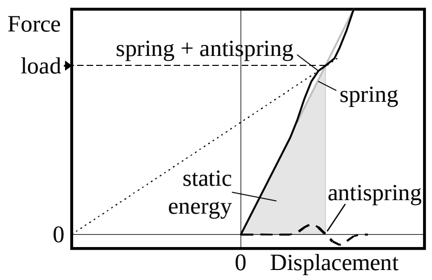

FIGURE 6. Force - displacement diagram for typical anti-spring system.

Since the mass of spring material used must be proportional to the total (static + dynamic) energy stored in the spring, it is clear from figure 6 that for a given resonant frequency, the spring mass used in such a non-linear system can be greatly reduced from its linear equivalent. It also becomes apparent that the best possible situation would be obtained if the static energy could be reduced to zero so that only dynamic energy storage need contribute to the spring mass. Remarkably this ideal is readily obtained by the very simple spring arrangement described hereafter.

3 Euler Buckling Spring

It is well known in engineering that a column of elastic material will support a load with virtually no deflection until at some critical value of load (dependent on its modulus and not on its yield strength) it suddenly starts to buckle. This is the sort of wall-shaped non-linearity (fig 9) that is required in the force-displacement characteristic to provide zero static energy. Figure 7 shows two cases of buckling columns - 7a having pinned ends (a pivot point at each end) and 7b having rigidly fixed or clamped ends.

The well known Euler column formula giving the critical load at buckling for the pin-ended case is $P_{cr}=\pi^2 EI/l^2$. Here E is modulus of elasticity, I is the area moment of inertia and l is the length of the column[9]. The critical load for the clamped end case is also well known and may be obtained by comparison :- It is apparent that the pin-ended column is the same curve as the centre section of the clamped end column between the points of inflection, which in this case is half the length. It follows that the critical load for a single clamped spring will be four times as great as for the same spring with pinned ends.

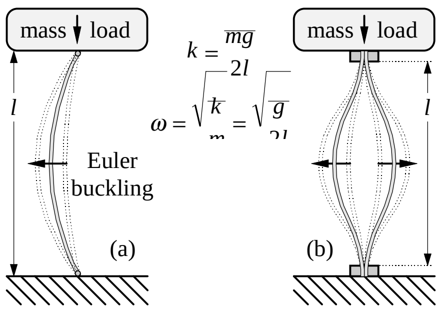

FIGURE 7. Spring blades in Euler column buckling mode.

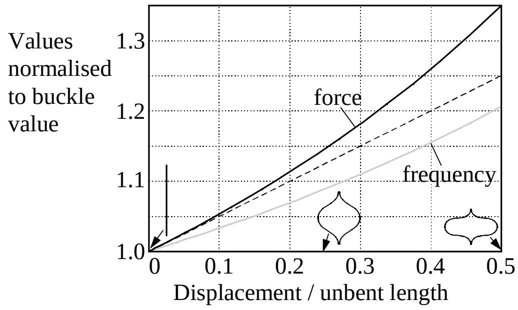

FIGURE 8. Force - displacement characteristic of elastic buckling.

What is not well known about Euler column buckling is the relationship of force vs displacement beyond the initial onset of buckling at critical load (as buckling is generally to be avoided!). The shape assumed by an inextensional elastically buckling spring is called an *elastica* and exact analysis of large deflections in such a spring involves the use of elliptic integrals. The application of elliptic integrals to the elastica is well covered in advanced texts[10] and used by ourselves in a related analysis of non-linear spring-rate reduction[11]. If the degree of buckling of a clamped spring as in figure 7b is specified by the maximum angle α_0 it attains with respect to its unbent centre line, then the force F (normalised to the critical load) and displacement x (normalised to the unbent spring length) are given by :-

$$F = 4\,K(k)^2/\pi^2$$
$$x = 2\big(1 - E(k)/K(k)\big) \qquad (1)$$

where $k=\sin(\alpha_0/2)$ is called the modulus and $K(k)$, $E(k)$ are complete elliptic integrals of the first and second kinds. These equations are plotted in figure 8 and show that the force vs displacement characteristic is remarkably well behaved with a low spring-rate and this remains reasonably constant even up to very large amounts of buckling.

Figure 9 illustrates that if the working range is designed to start just at the onset of buckling, then all of the energy stored by this type of spring is the dynamic energy exchanged in and out while operating within its working range. It also becomes apparent that the mass of spring required is directly proportional to the working range that can be accepted. For example suppose we can accept a working range of 0.5mm at a resonant frequency of 1Hz (normally requiring 25cm static extension in a linear system), then if made of the same elastic material the Euler spring may be 1/250th the mass of its linear equivalent. One would expect the internal resonant modes of the Euler spring to be roughly proportional to the square root of this mass ratio (ie $\sqrt{250} \approx 16$ times higher than the linear

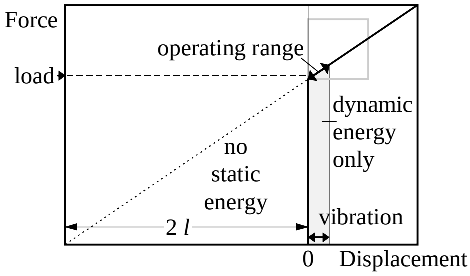

FIGURE 9. Force - displacement characteristic for Euler springs. Grey rectangle is expanded in figure 8.

case!) and the effect of the coupling at the internal resonances to be reduced by this mass ratio (ie 1/250th). In addition we will show (section 6) that spring rate reduction can also be achieved, to allow, in principle, even greater relative improvement.

4 Resonant Frequency

The ratio of the derivatives of equations (1) yield the spring-rate $k_s=\delta F/\delta x$ and this together with the mass $m=P_{cr}/g$ required to critically load the spring determine the resonant frequency $\omega=(k_s/m)^{½}$ of the system. Remarkably when an Euler spring as shown in figure 7 is critically loaded so that it just starts to buckle, then the vertical resonant frequency of the mass-spring system depends only on the length of the spring (and g = accel due to gravity). This might be expected from familiarity with the fact that a linear spring in vertical suspension and a pendulum share the same equation for resonant frequency - $\omega=(g/l)^{½}$ - with l being the extension of the linear spring under load or the length of the pendulum as the case demands. However for the Euler spring an extra factor of 2 is involved - the expression becomes $\omega_e=(g/2l)^{½}$ - so that the suspended mass moves as though it was suspended by a linear spring which had been extended by an amount *twice* the length of the Euler spring.

Figure 9 shows this initial spring-rate slope in the context of the critical load, while the grey rectangle is expanded in figure 8 to show in detail how the spring-rate and consequent resonant frequency vary with displacement. It should be noted that the displacement axis in figure 8 covers a very large travel range compressing the Euler springs to half their initial length (see spring shapes indicated), whereas a useful displacement (for minimising static energy etc) would typically compress by less than 1% of spring length. It is apparent that in theory the spring rate remains almost constant for such small displacements.

The resonant frequency relative to that at the start of buckling is also plotted and it can be seen that it only increases by 20% for a 50% compression of spring length.

5 Supporting Structures

In order to be useful as a suspension device the Euler springs need to be constrained within some structure to limit motion to the desired longitudinal compression. A simple mechanism to provide this constraint is the cantilever or pivoted lever arm shown in figure 10. The motion constraint in this case is the arc of a circle rather than a straight line, but it is approximately linear for the small working range required.

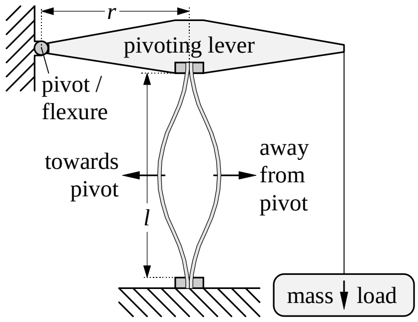

FIGURE 10. Simple structure for mounting and constraining Euler springs.

The slightly non-linear motion also provides some advantages. One is that the load can be supported or suspended at a different distance along the lever to that at which the spring blade is clamped - allowing a mechanical advantage ratio to be used to match an available spring to a particular load. This lever ratio may be adjustable to allow a continuous trade-off between supported mass and working range (mass × range = constant fixed by spring blade energy storage capacity). Another advantage if a bell-crank is employed (figure 11a) is that the spring blade can be mounted at any desired orientation around the pivot to make use of available space.

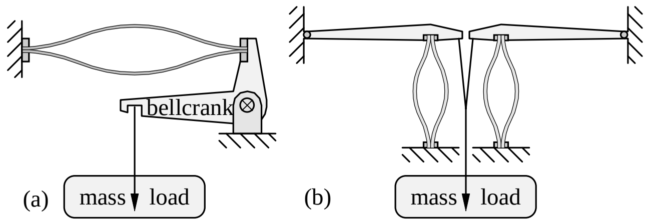

FIGURE 11. Additional structures for application of Euler springs.

However the main advantage is the ease with which spring-rate reduction techniques may be applied to rotational motion.

6 Spring-Rate Reduction

As a flat spring blade starts to buckle, the bulk of the blade can be offset in either one of two directions. If it is mounted in the pivoted support structure shown in figure 10, then the effect of offset in one direction is markedly different from the effect of offset in the other direction. If the offset occurs towards the pivot then a very low (and even negative = unstable) spring-rate is obtained (curve a in figure 12). If the offset occurs away from the pivot then a much higher spring-rate is obtained (12d). If a pair of matched spring blades are employed with one going in each direction, then the spring-rate (12c) is graphically indistinguishable from what it would have been if it had been constrained to move linearly rather than in the rotating support structure actually employed.

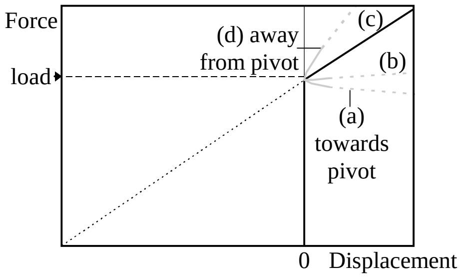

FIGURE 12**.** Force - displacement characteristic of individual springs.

It follows that by choosing an appropriate ratio between the bending stiffness of the blade(s) moving towards the pivot to those moving away, a suitable mix of 12a and 12d can be obtained giving a much reduced spring-rate 12b. This allows much lower resonant frequencies and also results in reduced Q-factor giving better damping. However a full analysis (to be published separately [11]) shows that all the curves apart from 12c are significantly non-linear and it is not so simple to reduce the spring-rate over a reasonable operating range with this method.

Figure 13 shows two spring rate reduction techniques which only use the linear case of pairs of Euler springs - one deflecting in each direction. Figure 13a has the load attached to what is in essence an inverse pendulum of height h positioned on top of the Euler springs. This structure has merit in that only very small forces are placed on the pivot allowing a lightweight structure and a very thin flexure to serve as the pivot. Taking the most simplistic view of a vertical force f acting in one suspension wire and from the Euler springs on one side, then this inverse pendulum produces an anti-rotational spring-rate of fh in the pivot. The normal rotational spring-rate due to the Euler springs is r^2 times the linear rate ie $r^2f/(2l)$. Equating these two expressions gives an approximation of the dimensions required to achieve spring-rate cancellation :- $h/r=½\ r/l$.

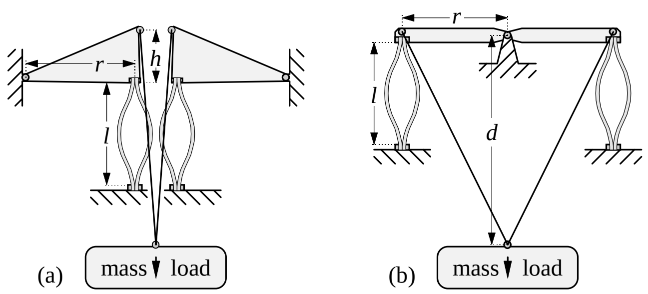

FIGURE 13**.** Spring-rate nulling in structures using pairs of Euler springs.

Figure 13b places one quarter of the total lifting force on the pivot and it is this horizontal force acting on the horizontal arm that produces the anti-spring effect. In this case the horizontal force is fr/h giving an anti-rotational spring-rate of fr^2/h. Equating this expression to the rotational spring-rate due to the Euler springs of $r^2f/(2l)$ gives dimensions for spring-rate cancellation of simply d=2l.

Techniques such as these which cancel almost constant spring-rates against each other, offer easily obtained low spring-rates with large operating range and low Q-factors. The structures that we have investigated (the two in figure 11) indicate that the predicted spring-rates near the buckling knee can only be achieved if great care is taken to fix the boundary conditions. Imperfect clamping rigidity and small offset angles causes the sharp corner discontinuity of figure 9 and 12 to be rounded into a gradual asymptotic approach to the theoretical spring-rate gradient at larger displacements. This can cause the spring-rate to be increased by an order of magnitude for small buckling displacements, giving a factor of 3 higher frequency than expected.

7 A High Performance Vertical Vibration Isolator

Initially a bell-crank version (very similar to figure 11a) was built which allowed fine adjustment of various parameters such as the clamping angles for each end of the Euler springs. This allowed many aspects of the theory to be verified[11] but the unbalanced structure with its strong coupling to tilt did not give a clean transfer function. So following this prototype, a higher performance balanced version similar to figure 11b and shown in figure 14 was built and tested.

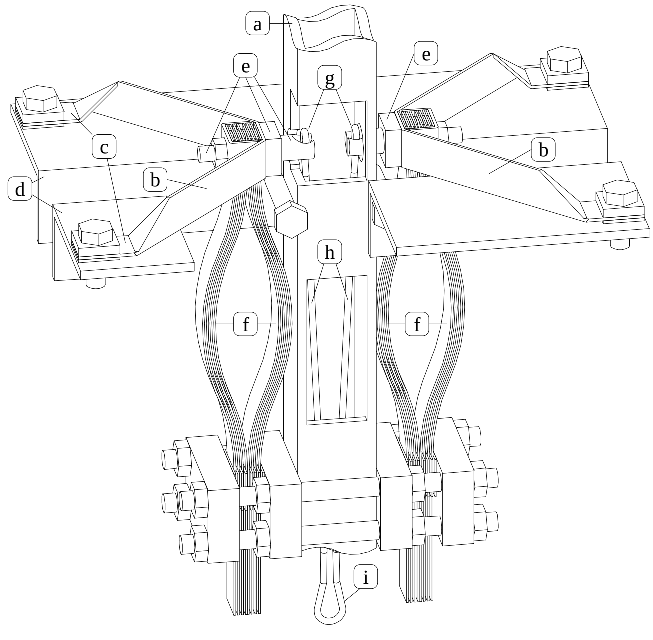

FIGURE 14**.** High performance vertical vibration isolator.

This particular implementation was designed as a single vertical isolation stage in a cascaded chain of stages hung vertically one below the other. The square tube (a) at the centre is the mounting platform for all the parts comprising the isolation stage (only the suspension components are shown and sections are cut away for viewing). The pivoting arms (b) are a “wishbone” shape for maximum rigidity and minimum mass. They are made of thin sheet and are intended to flex in the flat sections (c) near the clamp which can be thinned for this purpose. They are mounted on pieces of angle (d) clamped on to the central tube. There is a pair of special bolts (e) which pass through a hole in the ends of the spring blades (f) (made of feeler gauge stock) clamping a whole set of blades together and to the wishbones with spacers between each blade. The lower ends of the spring blades are clamped rigidly together with spacers to the central tube. The special bolts (e) clamping the upper ends of the blades pass through large clearance holes in the central tube to limit their motion to a safe operating range and contain a tapered socket to engage firmly with the folded over ends (g) of a loop of suspension wire (h). The suspended mass or following stages are hung by a pin through the centre of this loop of suspension wire (i).

8 Measured Results

The measurements shown here were for the structure of figure 14 but with a total of only two Euler spring blades (one each side bending outwards). Having blades only bending in the outward direction is normally unstable (fig 12 curve a) but can be stabilised by the contributing stiffness of the flexing wishbones (fig 14c) and bending wire (between g&h). In this case we made the wishbone flexing section (c) quite thick and made it adjustable in length using slots in the wishbone clamps. The spring blades were 0.8mm thick strips of feeler gauge with a length between clamps of 126mm and the system was loaded with a mass of 32kgs. It was designed to have a working range of ±1mm (ie 2mm total from unbuckled to motion limit).

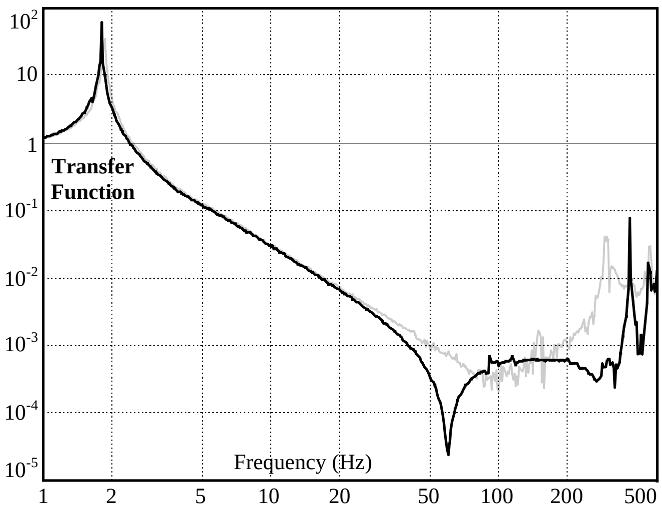

FIGURE 15. Transfer functions of Euler sprung vibration isolation stage.

A number of swept sine shake tests were done in air and a typical result is shown in black in figure 15. Here it can be seen that the first troublesome internal mode occurs around 400Hz. Evidence mentioned below indicates that this mode is the mass of the clamping bolts (e) resonating with the wire (h) and that the first blade internal mode is the small one appearing at almost 500Hz.

There is a clear notch at 60Hz and above that an isolation floor of -65 to -70dB. The notch is strongly suggestive of dynamic inertia (or centre of percussion) effects. To test this we loaded the wishbones (b) at about their mid point with small blocks of lead (of order 60g each side) with the resulting transfer function shown in grey. The notch at 60Hz disappeared as expected and a rounded shape falling to -70db around 100Hz was obtained confirming this suggestion. In addition the first internal mode resonance moved from 400Hz down to 300Hz indicating its dependence on the wishbone and clamping bolt mass, while the small mode near 500Hz remained unaltered. This indicated that the 400Hz mode was due to the wire and clamping bolts, rather than the Euler spring blades. Normally with very flexible joints at (c), this internal mode would not couple significantly, but in this particular case we had deliberately made the joints (c) very stiff to stabilise the negative spring-rate of the outward bending blades. The reason we were particularly interested in this configuration was because with the design of figure 14 it allows the outward bending blades (f) to be clamped very close to the tube (a) and wire (g), minimising forces in the wishbone (b).

9 Dynamic Effects

Normally with massive suspension structures, significant consideration must be given to the dynamic effect of the structure’s inertia (commonly called centre of percussion tuning)[3]. With the minimal spring mass to suspended mass ratio of this technique, it should be possible to neglect this effect. The transfer function in figure 15 shows that some improvement should be gained by appropriate counter-balancing (counter-weights between the 0g and 60g values used in figure 15 should give a lower transfer function floor level). However ideal counter-balancing is not possible and can only be approximated - due to the fact that the lateral motion of the spring mass is not proportional to compression but varies as its square root. This does not seem to be a significant disadvantage.

10 Spring Blade Variations

It is apparent that for a given working range, spring material yield stress, and mass to be supported, there are different selections of spring blades which may be applied. For example instead of a single pair of thick blades, two pairs of thinner and shorter blades may be used instead. It is of interest to know whether any advantage is to be gained by using more thin short blades rather than fewer thicker and longer blades.

As an example we may suppose that the thickness of a given pair of blades (eg feeler gauge strip 100mm long, 0.64mm thick giving a 1mm working range) are halved so that they may be bent to half the radius at the same stress level. To retain the same working range, the length of the thinner blades may only be reduced by a factor of 0.63. This increases the system resonant frequency by a factor of 1.26. The internal mode resonances scale as thickness/length^2 so in this case they will also increase by a factor of $0.5/0.63^2 = 1.26$. The suspension force decreases and therefore the number of blades must be increased by a factor of 3.17. A rule of thumb might be that if we halve the thickness of the blades, we can shorten them to 2/3 the length but must triple their width or number and we gain an increase in resonant frequency of internal modes of 25% but the ratio of internal mode to system resonant frequency remains unaltered.

It may have been noted, that the preceding discussions have only fully addressed the last of the three areas mentioned in section 2 - that of reducing the static energy and mass of the spring by producing an almost ideal wall-shaped non-linear force-displacement relation. There remains the possibility of improving performance further by (1) presetting some initial stresses within the springs, and (2a) moving some of the spring mass away from the centre of the Euler springs towards the ends where velocity of motion is less, or (2b) moving the spring mass closer to the centre of rotation as in a torsion based variant.

Presetting initial stresses may be done by starting with tightly rolled spring material, then unrolling it and bowing it somewhat the opposite way so that when released it lies flat. It must then be loaded in the bowed direction and this treatment should allow considerable extra deflection before yielding. This should allow a reduction in the mass of the springs by a factor of almost 2. We have not tried this, and neither have we explored any aspects of (2a) non-uniform Euler springs. However the gains available from (2b) moving the spring mass closer to a centre of rotation are obvious from a lever example :-

A very effective approach to obtain higher internal modes is to reduce the working range of the springs while retaining the required working motion of the stage with a lever as mentioned and illustrated in section 5. Halving the length, working range, and thickness of a blade also halves the force. If the working range is restored with a 2:1 lever, then this further halves the force requiring 4 times the number of blades to support the same load. However in this case the internal resonant frequencies are increased by a factor of 2 while the system resonant frequency remains unaltered. This impressive gain is the result of moving the spring mass closer to the centre of rotation of the lever.

11 Conclusion

Vertical isolation using mechanical springs (as opposed to compressed gas or magnetic suspension) has been plagued by the problem of internal modes bypassing the isolation at relatively low frequencies. The new technique presented here, which is most applicable for suspending constant loads under conditions of very small vibration, is capable of providing orders of magnitude better performance than previous approaches. Some remaining inefficiencies have been identified which may allow even better performance to be obtained from spring material of a given yield strength.

Acknowledgment

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