A method for the detection of gravitational waves from inspiralling compact binaries using a fast chirp transform

P R Charlton[1], S B Anderson[2], J K Blackburn[2], J Edlund[1], F A Jenet[3], A Lazzarini[2], T A Prince[1], M Tinto[1] and L Wen[1]

[1]Division of Physics, Mathematics and Astronomy, California Institute of Technology
[2]LIGO Laboratory, California Institute of Technology
[3]Schlumberger-Doll Research, Ridgefield CT 06877

Abstract. The Fast Chirp Transform (FCT) provides a powerful formalism for the detection of signals with variable frequency, in much the same way that the Fourier transform provides a formalism for the detection of constant frequency signals. The FCT algorithm has significant applications in the detection of gravitational waves, particularly the gravitational wave signal produced by an inspiralling compact binary. We report on progress towards a parallelised search code to identify inspiral events in interferometer data and estimate the mass parameters.

PACS numbers: 04.80.Nn, 02.30.Nw, 84.40.Ua, 95.75.Pq

1. Introduction

Matched filtering is a standard technique for the detection of signals in noisy data [1]. In the presence of coloured Gaussian noise $\langle n(t)\rangle$, we define the noise-weighted inner product of two real signals as

$$\langle u, v\rangle := 4\,\mathrm{Re}\int_0^\infty \frac{\tilde{u}(f)\,\tilde{v}_s^*(f)}{S(f)}\,df \tag{1}$$

where $\tilde{u}(f)$ denotes the Fourier transform of $u(t)$ and $S(f)$ is the one-sided noise power spectral density. Detection of a signal $h_s(t)$ in a data stream $h(t)$ is determined by the signal-to-noise ratio

$$\rho := \frac{\langle h, h_s\rangle}{\mathrm{rms}\,\langle n, h_s\rangle}. \tag{2}$$

When no signal is present, ρ is a random variable with Gaussian distribution and unit rms, thus when ρ exceeds a predetermined threshold we can state that a signal is present with known confidence.

Suppose that $h_s(t;\boldsymbol{\lambda})$ is a family of signal templates parametrised by $\boldsymbol{\lambda} \equiv \{\lambda_0, \lambda_1, \ldots, \lambda_P\}$, with $h_s(t;\boldsymbol{\lambda}) = 0$ for $t < 0$ and $t > T$. A signal with the same shape but commencing at time t_0 is given by $h_s(t - t_0; \boldsymbol{\lambda})$, thus when looking for a signal with unknown arrival time we wish to calculate

$$\rho(t_0, \boldsymbol{\lambda}) = \frac{4}{\sigma(\boldsymbol{\lambda})} \mathrm{Re} \int_0^\infty \frac{\tilde{h}(f)\, \tilde{h}_s^*(f; \boldsymbol{\lambda}) e^{-i2\pi t_0 f}}{S(f)}\, df \tag{3}$$

where $\sigma(\boldsymbol{\lambda}) = \mathrm{rms}\, \langle n, h_s(t; \boldsymbol{\lambda}) \rangle$. The discrete form of (3) is

$$\rho(k, \boldsymbol{\lambda}) = \frac{4}{N\sigma(\boldsymbol{\lambda})} \mathrm{Re} \sum_{n=0}^{N-1} \frac{\tilde{h}_n\, \tilde{h}_s^*(n/N; \boldsymbol{\lambda}) e^{-i2\pi kn/N}}{S_n} \tag{4}$$

2. The generalised FCT algorithm

The Fast Chirp Transform (FCT) algorithm [2] provides a compact and efficient means for calculating an approximation to (4) when $\tilde{h}_s(f; \boldsymbol{\lambda})$ is of the form

$$\tilde{h}_s(f; \boldsymbol{\lambda}) = A(f)\, e^{i2\pi(\lambda_0 \phi_0(f) + \lambda_1 \phi_1(f) + \cdots + \lambda_P \phi_P(f))} \tag{5}$$

Such waveforms appear in the stationary phase approximation to the Fourier transform of gravitational wave signals from inspiralling compact binaries [3].

A detailed derivation of the Fast Chirp Transform with a non-negative monotonically increasing phase function $\phi(f)$ may be found in [2]. In the following we outline a generalisation of the FCT for a non-negative but otherwise arbitrary phase function. From there it is trivial to generalise the FCT to any number of phase functions.

Suppose that $\phi(f)$ is non-negative and bounded function on the interval $[0, 1]$. Let

$$S_{kk_0} = \sum_{n=0}^{N-1} \tilde{h}_n e^{i2\pi(kn/N + k_0 \phi(n/N))}. \tag{6}$$

We wish to find an efficient way to calculate an approximation to (6). We do this by dividing the range of $\phi(f)$ into M equal "bins" and approximating each $\phi(n/N)$ by the nearest bin center. To simplify the exposition, we can assume without loss of generality that the phase function has been scaled to have a maximum of $(M-1)/M$, since any scaling of ϕ can be soaked up into the coefficient.

For M equal bins, the bin centers are m/M, and for each n, $\phi(n/N)$ can be approximated by $\mu(n)/M$ where

$$\mu(n) = \mathrm{round}\,[M\phi(n/N)] \tag{7}$$

so that

$$\left| \phi(n/N) - \frac{\mu(n)}{M} \right| \leq \frac{\Delta}{2} \tag{8}$$

where $\Delta = 1/M$ is the bin width. Note that although each $\mu(n)$ is an integer between 0 and $M-1$, the fact that $\phi(f)$ is arbitrary means that the integers $\mu(n)$ need not be distinct or have any special order.

Replacing each $\phi(n/N)$ in (6) by its approximation, we obtain

$$H_{kk_0} := \sum_{n=0}^{N-1} \tilde{h}_n e^{i2\pi(kn/N + k_0 \mu(n)/M)} \tag{9}$$

which is an approximation to S_{kk_0}. It is possible to express the summation H_{kk_0} as a 2-dimensional discrete Fourier transform by packing the input data $\tilde{h}_n$ into a sparse matrix in a certain way, which we now describe.

Given n between 0 and $N-1$, we know how to find m such that m/M is an approximation for $\phi(n/N)$ in the sense of (8). Now we turn the question around and ask: given m between 0 and $M-1$, for what integers n is $\phi(n/N)$ approximated by m/M? Let

$$N_m = \{n : n \in \{0,\ldots,N-1\} \text{ and } \mu(n) = m\}. \tag{10}$$

Then for each $n \in N_m$, $\phi(n/N)$ is approximated by $\mu(n)/M$. If, for example, $\phi(f)$ were monotonically increasing, each N_m would consist of a set of zero or more contiguous integers. For an arbitrary phase function, each N_m will be a (possibly empty) set of integers and will not generally be contiguous.

It is clear that the sets N_m partition $\{0,\ldots,N-1\}$, so we can sum over the union of all the N_m as easily as over N:

$$\begin{aligned} H_{kk_0} = & \sum_{n \in N_0} \tilde{h}_n e^{i2\pi(kn/N + k_0 0/M)} + \\ & \quad \sum_{n \in N_1} \tilde{h}_n e^{i2\pi(kn/N + k_0 1/M)} + \cdots \\ & \qquad \sum_{n \in N_{M-1}} \tilde{h}_n e^{i2\pi(kn/N + k_0 (M-1)/M)} \\ = & \sum_{m=0}^{M-1} \sum_{n \in N_m} \tilde{h}_n e^{i2\pi(kn/N + k_0 m/M)}. \end{aligned} \tag{11}$$

We can write (11) in a form resembling a 2-dimensional DFT by augmenting $\tilde{h}_n$ to a 2-dimensional sparse matrix. Let

$$\hat{h}_{nm} = \delta(\mu(n) - m)\tilde{h}_n. \tag{12}$$

For fixed m, as n varies it is clear that $\hat{h}_{nm}$ is zero when $n \notin N_m$, hence

$$H_{kk_0} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{h}_{nm} e^{i2\pi(kn/N + k_0 m/M)}. \tag{13}$$

This can be evaluated using the 2-dimensional FFT algorithm of order $O(MN \log MN)$, although a saving can be achieved by noticing that the data is arranged in $\hat{h}_{nm}$ so that column n has only a single non-zero element at row m, namely $\tilde{h}_n$. Effectively, the non-zero elements of $\hat{h}_{nm}$ lie on a "track" which is a discretisation of the graph of $\phi(f)$. Since the order of the summations in the 2-D DFT is unimportant, we can compute (13) by doing the N 1-D Fourier transforms in the M direction first. Since these are Fourier transforms of sequences with only one non-zero value, they can be done directly as N $O(M)$ operations rather than using an FFT. The M 1-D Fourier transforms in the N direction are then calculated via the FFT, so the dominant cost of (13) is $O(MN \log N)$.

3. Application to detection of binary inspiral chirps

To illustrate the use of the FCT in gravitational wave detection, we discuss the post-Newtonian expansion for gravitational waves emitted during the binary inspiral of neutron stars and black holes. In laser interferometric detection of such signals, the SNR is obtained from (3), where $\tilde{h}(f)$ is the Fourier transform of the signal generated from the differential output of the interferometer, $\tilde{h}_s(f;\boldsymbol{\lambda})$ are Fourier transforms of the theoretical time-domain binary inspiral waveforms, and $S(f)$ is the average measured power spectral density of the interferometer output [4, 5].

The Fourier transform of the binary inspiral signal is most easily obtained via the stationary phase approximation [3]. Truncated to first post-Newtonian order it has the form

$$\tilde{h}_s(f;\mathcal{M},\eta) = f^{-7/6} e^{i(-\pi/4-2\phi_c+2\pi t_c f+\Psi(f;\mathcal{M},\eta))}, \tag{14}$$

$$\Psi(f;\mathcal{M},\eta) = \frac{3}{128\eta}\left[(\pi\mathcal{M}T_\odot f)^{-5/3} + \left(\frac{3715}{756}+\frac{55}{9}\eta\right)(\pi\mathcal{M}T_\odot f)^{-1}\right] \tag{15}$$

where t_c is the time of coalescence, ϕ_c is the phase at coalescence, $\mathcal{M} = m_1 + m_2$ is the total mass of the two bodies as a fraction of solar mass $\mathcal{M}_\odot$, and $\eta = m_1 m_2/\mathcal{M}^2$. The constant $T_\odot = G\mathcal{M}_\odot/c^3$ has a value of approximately 4.925×10^{-6}s.

It is convenient to replace the mass parameters $\mathcal{M}$ and η by the chirp times

$$\tau_0 = \frac{5}{256}\mathcal{M}^{-5/3}\eta^{-1}(\pi f_0)^{-8/3}, \tag{16}$$

$$\tau_1 = \frac{5}{192}\mathcal{M}^{-1}\left(\frac{743}{336\eta}+\frac{11}{4}\right)(\pi f_0)^{-2} \tag{17}$$

which are the Newtonian and first post-Newtonian contributions to the time it takes the instantaneous frequency to evolve from f_0 to infinity [6]. The reference frequency f_0 is chosen to be the frequency at which the noise power $S(f)$ is a minimum. For LIGO I we take $f_0 = 175$ Hz. With this substitution we have

$$\Psi(f;\tau_0,\tau_1) = \frac{6}{5}\pi\tau_0 f_0\left(\frac{f}{f_0}\right)^{-\frac{5}{3}} + 2\pi\tau_1 f_0\left(\frac{f}{f_0}\right)^{-1}. \tag{18}$$

Apart from the leading constants in the phase, $\tilde{h}_s(f;\tau_0,\tau_1)$ is thus of the form (5).

An FCT with two phase functions can be used to approximate the discrete version of the matched filter output (3) for this waveform. The phase functions in (18) are bounded by introducing a low frequency cutoff f_s below which the signal is set to zero. For LIGO I we use $f_s = 40$ Hz, the point at which seismic noise begins to dominate the noise spectrum. Up to an overall phase factor depending on ϕ_c, the FCT approximation to the discrete form of (3) is

$$H_{kk_0k_1} = \sum_{l=0}^{L-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{h}_{nml} e^{i2\pi(kn/N+k_0m/M+k_1l/L)} \tag{19}$$

where

$$\hat{h}_{nml} = \delta(\mu_0(n)-m)\,\delta(\mu_1(n)-l)\tilde{h}_n, \tag{20}$$

$$\mu_0(n) = \mathrm{round}\left[M\phi_0(n/N)\right], \tag{21}$$

$$\mu_1(n) = \mathrm{round}\left[L\phi_1(n/N)\right] \tag{22}$$

the phase functions have been scaled so that

$$\phi_0(f) = \frac{M-1}{M}\left(\frac{f}{f_s}\right)^{-\frac{5}{3}},\ f \geq f_s,\ \text{otherwise } 0 \tag{23}$$

$$\phi_1(f) = \frac{L-1}{L}\left(\frac{f}{f_s}\right)^{-1},\ f \geq f_s,\ \text{otherwise } 0 \tag{24}$$

and the input data $\tilde{h}_n$ to the FCT has been precalculated as the product of the Fourier transform of the interferometer output, the amplitude of $\tilde{h}_s(f;\tau_0,\tau_1)$ and the inverse of the noise power. The relationship between the indices k_0, k_1 and the chirp times is

$$\tau_0 = \frac{5}{3}\frac{M}{M-1} f_0^{-1}\left(\frac{f_s}{f_0}\right)^{\frac{5}{3}} k_0, \tag{25}$$

$$\tau_1 = \frac{L}{L-1} f_0^{-1}\left(\frac{f_s}{f_0}\right) k_1. \tag{26}$$

4. Parallel implementation

From (19)–(22) it is clear how the FCT can be generalised to any number of phase functions. The most direct implementation is to pack the data h_{n_0} into $\hat{h}_{n_0 n_1 \ldots n_P}$ as in (22) and perform a full P-dimensional FFT of $\hat{h}_{n_0 n_1 \ldots n_P}$. However, as in the case of the FCT for one phase function, the data h_{n_0} is arranged so that it follows a one-dimensional "track" in $\hat{h}_{n_0 n_1 \ldots n_P}$, where there is only a single non-zero entry in each $(P-1)$-dimensional matrix obtained by fixing n_0. Thus the P-dimensional FFT can be evaluated by performing a trivial $(P-1)$-dimensional DFT for each n_0 then performing a 1-D FFT for each $n_1, n_2, \ldots, n_P$, that is, $N_1 \times N_2 \times \ldots \times N_P$ 1-dimensional FFT's in the N_0 direction. A single-processor implementation of this method of calculating the FCT for any number of phase functions is available in the LIGO Analysis Library (LAL) [7].

While the LAL FCT implementation itself is not parallel, it has proven simple to parallelise it by incorporating it into the general-purpose Message Passing Interface (MPI) wrapper for parallel code developed in the LIGO Data Analysis System (LDAS) [8]. The LAL FCT implementation permits a range of values to be chosen for each chirp index $\{k_0, k_1, \ldots, k_P\}$, enabling the calculation of a partial FCT. In effect, the full 1-D FFT's are calculated in the N_0 direction only for the specified values of $\{k_0, k_1, \ldots, k_P\}$. This is equivalent to approximating the optimal filter outputs for a range of discretely chosen parameters.

The LDAS MPI wrapper provides a simple interface to MPI which allows single-processor code to run in parallel on multiple nodes of a Beowulf cluster. In parallelising the FCT, each node is assigned a partial FCT to calculate for some range of the chirp indices (equivalently, some region of chirp parameter space) for the same input data h_{n_0}. The MPI wrapper handles most of the MPI management, including starting processes on different processors and propagating data to each of them. The user code run by the wrapper is simply single-processor code, with the addition of a few MPI function calls to manage synchronisation and communication. The parallel FCT code was successfully tested during a joint Mock Data Challenge with the Inspiral Upper Limits Group in May 2001 and is now available as part of the `lalwrapper` package [9]. This MDC verified the functional integrity of the parallel FCT using signals with known parameters injected into white noise at a comparatively high signal-to-noise ratio. The next stage of development is scientific verification via detection of simulated signals with unknown parameters using simulated LIGO noise model.

Acknowledgments

The authors would like to thank Patrick Brady, Duncan Brown and Maria Barnes for their help with MPI coding issues.

This work has been partially supported by NSF grant PHY00-71050. LIGO Laboratory is supported by the NSF under cooperative agreement PHY92-10038.

This publication has been assigned the LIGO document number LIGO-P010021-00-E.

References

[1] Helstrom C 1968 *Statistical Theory of Signal Detection* (Pergamon, Oxford)
[2] Jenet F A and Prince T A 2000 Detection of variable frequency signals using a fast chirp transform *Phys. Rev. D* **62** 122001
[3] Droz S, Knapp D, Poisson E and Owen B *Phys. Rev. D* **59** 124016
[4] Thorne K 1987 in *300 Years of Gravitation* ed Israel W and Hawking S W (Cambridge University Press) 330–458
[5] Schutz B 1991 in *The Detection of Gravitational Waves* ed Blair D (Cambridge University Press) 406–452
[6] Owen B and Sathyaprakash B S 1999 *Phys. Rev. D* **60** 022002
[7] LIGO/LSC Algorithm Library `www.lsc-group.phys.uwm.edu/lal`
[8] Blackburn K *et al.* The wrapper API baseline requirements and implementation *LIGO Technical Document* T990097-14-E
[9] LIGO/LSC MPI Working Group `www.lsc-group.phys.uwm.edu/mpigroup`